# Experimental Number Theory
# Part I : Tower Arithmetic

by Edinah K. Gnang

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## 1 Introduction

We introduce in this section an Algebraic and Combinatorial approach to the theory of Numbers. The approach rests on the observation that numbers can be identified with familiar combinatorial objects namely *rooted trees*, which we shall here refer to as *towers*. The bijection between numbers and towers provides some insights into unexpected connexions between Number theory, combinatorics and discrete probability theory.

### Definition 1.1

Let $\boldsymbol{X}$ denote a $n$ dimensional vector whose entries are distinct variables defined by

$$\boldsymbol{X} = (x_k)_{1 \leq k \leq n}, \tag{1}$$

a *tower expansion* ( or simply a tower ) over $\boldsymbol{X}$ is a finite product of iterated exponentiations over the entries of $\boldsymbol{X}$. Furthermore the set of towers over the entries of $\boldsymbol{X}$ is denoted $\mathcal{T}(\boldsymbol{X})$.

### Example 1.1

Let $\boldsymbol{X} = (x, y)$, the expressions bellow feature three towers

$$x, \quad x^{(y^x)}, \quad x^{(y^x) \cdot \left( x^{(x^y)} \right)} \cdot y^{(x^y) \cdot \left( y^{(y^y)} \right)}. \tag{2}$$

The *height* of a tower[1] indicates the maximum number of iterated exponentiations occurring in the tower, the *base* of the tower refers to the bottom level of the tower. Finally we will call the variables appearing at the base level of a tower the *pillars* of the tower.

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[1] I shall often omit the parenthesis indicating the order in which the iterated exponentiations are being carried out, but we shall always assume that the iterations are performed from the top down .

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***Theorem***: **Fundamental Theorem of Arithmetic ( F. T. A. ):** Every positive integer greater than 1 can be written uniquely as a product of powers of primes. ( the expression written in non decreasing order of the primes.)

***Corollary*** : Every positive integer greater than 1 can be written uniquely as a tower expansion over the primes ( the primes at each level of the tower are written in increasing order ).
we will not discuss here the proof of the F. T. A. but refer the reader to a beautiful discussion on the proof of the F. T. A. in [1].

### Definition 1.2

A *Formal Tower Series* is a series expansion which consists of a linear combination of distinct but not necessarily finitely many towers. The coefficients in the linear combination are assumed to originate from a field noted here $\mathbb{F}$ ( preferably finite ). The set of such Formal Tower Series is denoted $\mathbb{F}\left[\mathcal{T}\left(\boldsymbol{X}\right)\right]$. Furthermore a linear combination of finitely many distinct towers will be called a *polytower*.

## 1.1   Revisiting Euler's product formula for the Riemann zeta function.

Let $\boldsymbol{X}$ denote an infinite dimensional vector of variables defined by

$$\boldsymbol{X} = (x_k)_{1 \leq k \leq \infty} \tag{3}$$

As mentioned earlier the F. T. A. induces a bijection between $\mathbb{N}\setminus\{0,1\}$ and towers over the vector of primes, that bijection in turn suggests a natural bijection between $\mathbb{N}\setminus\{0,1\}$ and $\mathcal{T}\left(\boldsymbol{X}\right)$ as illustrated bellow

$$
\begin{aligned}
T_{\boldsymbol{X}}\left(2\right) &= & x_1 \\
T_{\boldsymbol{X}}\left(3\right) &= & x_2 \\
T_{\boldsymbol{X}}\left(4\right) &= & x_1^{x_1} \\
T_{\boldsymbol{X}}\left(5\right) &= & x_3 \\
T_{\boldsymbol{X}}\left(6\right) &= & x_1\,x_2 \\
T_{\boldsymbol{X}}\left(7\right) &= & x_4 \\
&\vdots&
\end{aligned}
\tag{4}
$$

Let us now introduce the binary operator $\mathfrak{R}\left(.\,,\,.\right)$ which we will refer to as the *raiser operator* for reason that will be apparent subsequently.

$$\mathfrak{R}:\ \{T_{\boldsymbol{X}}\left(k\right)\}_{1 \leq k \leq \infty} \times \mathbb{F}\left[\mathcal{T}\left(\boldsymbol{X}\right)\right] \rightarrowtail \mathbb{F}\left[\mathcal{T}\left(\boldsymbol{X}\right)\right]$$

$$\mathfrak{R}\left(x,\ \sum_{k\in\mathbb{N}/\{0,1\}} a_k\,T_{\boldsymbol{X}}\left(k\right)\right) = \sum_{k\in\mathbb{N}\setminus\{0,1\}} a_k\,x^{T_{\boldsymbol{X}}\left(k\right)}\,. \tag{5}$$

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More generally we write

$$\mathfrak{R} : \{T_{\boldsymbol{X}}(k)\}_{1 \leq k \leq \infty} \times \mathbb{F}\left[\mathcal{T}\left(\boldsymbol{X}\right)\right] \rightarrowtail \mathbb{F}\left[\mathcal{T}\left(\boldsymbol{X}\right)\right]$$

$$\mathfrak{R}\left(T_{\boldsymbol{X}}(l), \sum_{k \in \mathbb{N}/\{0,1\}} a_k\, T_{\boldsymbol{X}}(k)\right) = \sum_{k \in \mathbb{N}\backslash\{0,1\}} a_k\, \left(T_{\boldsymbol{X}}(l)\right)^{T_k(\boldsymbol{X})}. \qquad (6)$$

For most of our discussion however we will require the first of the two definition of *raiser operator* but we point out that the first definition follows from the more general following definition.

We recall Euler's product identity for the Riemann Zeta function as expressed by

$$\prod_{k \in \mathbb{N}\backslash\{0\}} \left(1 - p_k^{-s}\right)^{-1} = 1 + \sum_{n \in \mathbb{N}\backslash\{0,1\}} n^{-s}. \qquad (7)$$

We will come to think of the identity above as expressing an invariance principle. One important reason for introducing formal tower series is to validate in some sense identities of the form

$$\prod_{k \in \mathbb{N}\backslash\{0\}} \left(\sum_{t \in \mathbb{N}} p_k^t\right) = 1 + \sum_{n \in \mathbb{N}\backslash\{0,1\}} n \qquad (8)$$

Here is how the expression above can be thought to be not only meaningful but also depicting a fundamental invariance principle.

$$\prod_{k \in \mathbb{N}\backslash\{0\}} \left(1 + \mathfrak{R}\left(x_k,\, 1 + \sum_{n \in \mathbb{N}\backslash\{0,1\}} T_{\boldsymbol{X}}(n)\right)\right) = 1 + \sum_{n \in \mathbb{N}\backslash\{0,1\}} T_{\boldsymbol{X}}(n) \qquad (9)$$

so that the Formal Tower Series $\left(1 + \sum_{n \in \mathbb{N}\backslash\{0,1\}} T_{\boldsymbol{X}}(n)\right)$ is invariant under the action of the operator

$$\prod_{k \in \mathbb{N}\backslash\{0\}} \left(1 + \mathfrak{R}\left(x_k,\, \centerdot\right)\right) \qquad (10)$$

To get a sense of how such an invariance principle could naturally arises we consider the function.

$$g(x) = 1 + x + x^x + x^{x^x} + x^{x^{x^x}} + \cdots \qquad (11)$$

and use it to induce a sequence of functions on the vector $\boldsymbol{X}$ who's initial element is

$$G_0\left(\boldsymbol{X}\right) = \prod_{1 \leq k \leq \dim\{\boldsymbol{X}\}} \left(1 + g\left(x_k\right)\right) \qquad (12)$$

and the other elements of the sequence are defined by the following recursion

$$G_{n+1}\left(\boldsymbol{X}\right) = \left(\prod_{1 \leq k \leq \dim\{\boldsymbol{X}\}} \left(1 + \mathfrak{R}\left(x_k, G_n\left(\boldsymbol{X}\right)\right)\right)\right) \qquad (13)$$

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So that the fundamental invariance principle is re-casted as

$$\lim_{n \to \infty} \{G_n(\boldsymbol{X})\} = 1 + \sum_{n \in \mathbb{N} \setminus \{0,1\}} T_{\boldsymbol{X}}(n) \tag{14}$$

The invariance principle follows from the F. T. A where $\boldsymbol{X}$ will represent the vector whose entries are the distinct primes arranged in increasing order[2].

Algorithms such as Buchberger's algorithm in commutative algebra emphasizes the importance of totally ordering monomials. In our discussion we shall use the integer ordering to induce a natural ordering on the towers. Once the towers are totally ordered it becomes rather straight forward to discuss *Tower Arithmetic*. Let us encapsulate the ordering of towers into a metric function $d(\boldsymbol{.},\boldsymbol{.})$ so as to embed towers into a metric space $(\mathcal{T}(\boldsymbol{X}), d(\boldsymbol{.},\boldsymbol{.}))$. The metric space $(\mathcal{T}(\boldsymbol{X}), d(\boldsymbol{.},\boldsymbol{.}))$ allows us express addition of towers through the following relation

$$for \quad T_{\boldsymbol{X}}(p) \geq T_{\boldsymbol{X}}(m)$$

$$d(T_{\boldsymbol{X}}(m), T_{\boldsymbol{X}}(p)) = d(0, T_{\boldsymbol{X}}(n)) \Leftrightarrow T_{\boldsymbol{X}}(m) + T_{\boldsymbol{X}}(n) = T_{\boldsymbol{X}}(p). \tag{15}$$

and we use the convention

$$T_{\boldsymbol{X}}(0) = 1 \quad and \quad T_{\boldsymbol{X}}(0) + T_{\boldsymbol{X}}(0) = x_1 \tag{16}$$

Furthermore multiplication of tower follows immediately from the definition of addition and it is expressed by

$$\left( \prod_{1 \leq k \leq \dim\{\boldsymbol{X}\}} x_k^{T_{\boldsymbol{X}}(m_k)} \right) \times \left( \prod_{1 \leq t \leq \dim\{\boldsymbol{X}\}} x_t^{T_{\boldsymbol{X}}(n_t)} \right) =$$

$$\left( \prod_{1 \leq k \leq \dim\{\boldsymbol{X}\}} x_k^{(T_{\boldsymbol{X}}(m_k) + T_{\boldsymbol{X}}(n_k))} \right) \tag{17}$$

In summary the base level of the product of tower is the union of the base level of the towers being multiplied while the powers of corresponding pillars are added.

We now consider the case of finite dimensional vectors. Let $\boldsymbol{P}$ be a finite dimensional vector whose entries are made of the smallest $\dim\{\boldsymbol{P}\}$ distinct primes. For convenience we arrange the primes in increasing order as entries of $\boldsymbol{P}$ we have

$$\boldsymbol{P} = (p_1, \cdots, p_{\dim\{\boldsymbol{P}\}}) \tag{18}$$

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[2]the ordering is not necessary for the invariance principle it suffice to have distinct the entries in $\boldsymbol{X}$ of must corresponding to distinct primes.

we define an analogous sequence of functions of $\boldsymbol{P}$ defined by

$$G_0\left(\boldsymbol{P}\right) = \prod_{1 \leq k \leq \dim\{\boldsymbol{P}\}} \left(1 + g\left(p_k\right)\right) \tag{19}$$

and the recursion

$$G_{n+1}\left(\boldsymbol{P}\right) = \prod_{1 \leq k \leq \dim\{\boldsymbol{P}\}} \left(1 + \mathfrak{R}\left(p_k,\ G_{n+1}\left(\boldsymbol{P}\right)\right)\right) \tag{20}$$

in which case we obtain

$$\lim_{n \to \infty} \left\{G_n\left(\boldsymbol{P}\right)\right\} = 1 + \sum_{k \in \mathbb{N}/\{0,1\}} a_k\, T_{\boldsymbol{P}}\left(k\right) \tag{21}$$

where $a_k \in \{0, 1\}$, more specifically $a_k = 1$ if the tower expansion of $n$ is a tower over $\boldsymbol{P}$ and $a_k = 0$ otherwise.

The preceding discussion raises the following interesting question: Considering a given finite set of consecutive of integers bounded by $n$. What is the probability that a number chosen at random contains a particular prime $p$ in it's tower expansion

We now propose a theorem which follows from Euler's argument in his proof of the Infinity of the primes.

***Theorem*** : For every finite dimensional vector $\boldsymbol{P} = \left(p_1, \cdots, p_{\dim\{\boldsymbol{P}\}}\right)$ whose entries are made up of distinct primes when considering the limit

$$\lim_{n \to \infty} \left\{G_n\left(\boldsymbol{P}\right)\right\} = 1 + \sum_{n \in \mathbb{N}/\{0,1\}} a_n\, T_{\boldsymbol{P}}\left(n\right). \tag{22}$$

we have

$$\left(1 + \sum_{n \in \mathbb{N}/\{0,1\}} a_n\, \left(T_{\boldsymbol{P}}\left(n\right)\right)^{-1}\right) < \infty \tag{23}$$

**Proof** : The convergence follows immediately from the fact that

$$\left(1 + \sum_{n \in \mathbb{N}/\{0,1\}} a_n\, \left(T_{\boldsymbol{P}}\left(n\right)\right)^{-1}\right) < \left(\prod_{1 \leq k \leq \dim\{\boldsymbol{P}\}} \left(1 - p_k^{-1}\right)\right) < \infty \tag{24}$$

The preceding theorem suggests that the rational numbers must not be too far out of our reach once we are equipped with a concrete description of the

integers as towers. we recall that for a vector $\boldsymbol{P} = \left(p_1, \cdots, p_{\dim\{\boldsymbol{P}\}}\right)$ whose entries are made up of distinct primes. We consider the sequence

$$G_0\left(\boldsymbol{P}\right) = \prod_{1 \leq k \leq \dim\{\boldsymbol{P}\}} \left(1 + g\left(p_k\right)\right) \tag{25}$$

$$G_{n+1}\left(\boldsymbol{P}\right) = \prod_{1 \leq k \leq \dim\{\boldsymbol{P}\}} \left(1 + \mathfrak{R}\left(p_k,\, G_n\left(\boldsymbol{P}\right)\right)\right) \tag{26}$$

This sequence may be used to induce the sequence $H_n$ defined by

$$H_n\left(\boldsymbol{P}\right) = \prod_{1 \leq k \leq \dim\{\boldsymbol{P}\}} \left(\mathfrak{R}\left(p_k{}^{-1},\, G_n\left(\boldsymbol{P}\right)\right) + 1 + \mathfrak{R}\left(p_k, G_n\left(\boldsymbol{P}\right)\right)\right) \tag{27}$$

So that the terms in the resulting expression are given by

$$\mathrm{Lim}_{n \to \infty}\left\{H_n\left(\boldsymbol{P}\right)\right\} = 1 + \sum_{q \in \mathbb{Q}\backslash\{0,1\}} a_q\, T_{\boldsymbol{P}}\left(q\right). \tag{28}$$

If we seek the complete bijection with the rational we would have started with the infinite dimensional vector $\boldsymbol{X}$ instead and considered the following sequence

$$G_0\left(\boldsymbol{X}\right) = \prod_{1 \leq k \leq \dim\{\boldsymbol{X}\}} \left(1 + g\left(x_k\right)\right) \tag{29}$$

$$G_{n+1}\left(\boldsymbol{X}\right) = \prod_{1 \leq k \leq \dim\{\boldsymbol{X}\}} \left(1 + \mathfrak{R}\left(p_k,\, G_n\left(\boldsymbol{X}\right)\right)\right) \tag{30}$$

This sequence may be used to induce the sequence $H_n$ defined by

$$H_n\left(\boldsymbol{X}\right) = \prod_{1 \leq k \leq \dim\{\boldsymbol{X}\}} \left(\mathfrak{R}\left(p_k{}^{-1},\, G_n\left(\boldsymbol{X}\right)\right) + 1 + \mathfrak{R}\left(p_k,\, G_n\left(\boldsymbol{X}\right)\right)\right) \tag{31}$$

towers in bijections set $\mathbb{Q}\backslash\{0,1\}$ are described by

$$\mathrm{Lim}_{n \to \infty}\left\{H_n\left(\boldsymbol{P}\right)\right\} = 1 + \sum_{q \in \mathbb{Q}\backslash\{0,1\}} T_{\boldsymbol{P}}\left(q\right). \tag{32}$$

## 1.2  Tower Sieve Algorithm

Sieves play an important role in Number theory, We propose to investigate here a novel sieve algorithm based on the arithmetic of towers. Let $\boldsymbol{P} = \left(p_1, \cdots, p_{\dim\{\boldsymbol{P}\}}\right)$ denote vector whose entries are all the primes less the $2^t$ for some $1 \leq t$, the goal is to determine the primes in the range $\left[2^t, 2^{1+t}\right]$. The algorithm consists in computing the following recursion

$$g_s(x) = 1 + x + x^x + \cdots + \left(x^{x^{\cdot^{\cdot^{\cdot^x}}}} \right\} \text{last term of height } s\right) \tag{33}$$

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$$G_0\left(\boldsymbol{P}\right) = \prod_{1 \leq k \leq \dim\{\boldsymbol{P}\}} g_{s_k}\left(p_k\right) \tag{34}$$

where $p_k^{p_k^{\cdot^{\cdot^{p_k}}}}$ is the last term in the expression $g_{s_k}\left(p_k\right)$ is such that $p_k^{p_k^{\cdot^{\cdot^{p_k}}}} \ll 2^{1+t}$

$$G_{n+1}\left(\boldsymbol{P}\right) = \left(\prod_{1 \leq k \leq \dim\{\boldsymbol{P}\}} \left(1 + \mathfrak{R}\left(p_k, \, G_n\left(\boldsymbol{P}\right)\right)\right)\right) \tag{35}$$

we stop the recursion at the $m^{\text{th}}$ iteration if all the towers remaining in the polytower difference $\left(G_{m+1}\left(\boldsymbol{P}\right) - G_m\left(\boldsymbol{P}\right)\right)$ are are towers greater than $2^{1+t}$.

At the heart of the recursive algorithm is the fact that the recursion determines the tower expansions of integers in the interval $\left[2^t, \, 2^{1+t}\right]$ with the exception of towers expansion which contains primes which are not less than $2^t$. Furthermore assuming that we order the towers the gaps of size 2 in the list determines the exact location of primes in the range of interest. This provide a constructive proof of the fact that there is always at least one prime in the range $\left[2^t, \, 2^{1+t}\right]$ for any value of $1 \leq t$. The sieves algorithm we discussed above is rather different from Eratosthenes sieve in that it generates the composite and indicates exactly where the primes ought to be found and more importantly as oppose to some variants of Eratosthenes sieve methods our algorithm ensures that each composite is generated exactly once. Let us briefly go through the steps the algorithm with the case $t = 2$ for illustration purposes

## 1.3 Illustration of the Algorithm

We illustrate the algorithm using Mathematica.
Let $\boldsymbol{X} = (x_1, x_2)$ the Mathematica commands used are

$$g_8 := 1 + x_k$$

$$\textbf{For}\left[i = 1, i < 7, i{+}{+}, g_8 = \left(1 + \textbf{Total}\left[\left(x_k{}^{\wedge}\textbf{List@@}(g)\right)\right]\right)\right]$$

from which $g_8$ is given by

$$\boldsymbol{g} = 1 + x_k + x_k^{x_k} + x_k^{x_k^{x_k}} + x_k^{x_k^{x_k^{x_k}}} + x_k^{x_k^{x_k^{x_k^{x_k}}}} + x_k^{x_k^{x_k^{x_k^{x_k^{x_k}}}}} + x_k^{x_k^{x_k^{x_k^{x_k^{x_k^{x_k}}}}}} \tag{36}$$

The Mathematica commands for the recursion are given by

$$G_0 := 1$$

$$\textbf{For}\left[k = 1, k < 3, k{+}{+}, G_0 = \textbf{Expand}\left[G_0\left(1 + x_k\right)\right]\right]$$

after these first commands we have

$$\boldsymbol{G_0} = 1 + x_1 + x_2 + x_1 x_2 \tag{37}$$

Here is an overview of the typical intermediary steps required to compute the recursion.

$$\textbf{List@@}\left(\boldsymbol{G_0}\right) = \{1, x_1, x_2, x_1 x_2\}$$

$$\{\boldsymbol{1, x_1, x_2, x_1 x_2}\} / .\boldsymbol{x_1 \to 2} / .\boldsymbol{x_2 \to 3} = \{1, 2, 3, 6\}$$

$$\left(\boldsymbol{x_1}\textbf{\^{}List@@}\left(\boldsymbol{G_0}\right)\right) = \{x_1, x_1^{x_1}, x_1^{x_2}, x_1^{x_1 x_2}\}$$

$$\textbf{Total}\left[\left(\boldsymbol{x_1}\textbf{\^{}List@@}\left(\boldsymbol{G_0}\right)\right)\right] = x_1 + x_1^{x_1} + x_1^{x_2} + x_1^{x_1 x_2}$$

$$\textbf{Expand}\left[\left(1 + \textbf{Total}\left[\left(\boldsymbol{x_1}\textbf{\^{}List@@}\left(\boldsymbol{G_0}\right)\right)\right]\right)\left(1 + \textbf{Total}\left[\left(\boldsymbol{x_2}\textbf{\^{}List@@}\left(1 + \boldsymbol{x_1}\right)\right)\right]\right)\right]$$

$$= 1 + x_1 + x_1^{x_1} + x_1^{x_2} + x_1^{x_1 x_2} + x_2 + x_1 x_2 + x_1^{x_1} x_2 + x_1^{x_2} x_2 +$$

$$x_1^{x_1 x_2} x_2 + x_2^{x_1} + x_1 x_2^{x_1} + x_1^{x_1} x_2^{x_1} + x_1^{x_2} x_2^{x_1} + x_1^{x_1 x_2} x_2^{x_1}$$

The polytower contains the towers of interest and the list bellow depicts the corresponding numbers.

$$\textbf{L0} = \textbf{Sort}[[\textbf{List@@Expand}\left[\left(1 + \textbf{Total}\left[\left(\boldsymbol{x_1}\textbf{\^{}List@@}\left(\textbf{Expand}\left[\left(1 + \boldsymbol{x_1}\right)\left(1 + \boldsymbol{x_2}\right)\right]\right)\right)\right]\right)\right.$$

$$\left.\left(1 + \textbf{Total}\left[\left(\boldsymbol{x_2}\textbf{\^{}List@@}\left(1 + \boldsymbol{x_1}\right)\right)\right]\right)\right] / .\boldsymbol{x_1 \to 2} / .\boldsymbol{x_2 \to 3}]$$

$$= \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, 64, 72, 192, 576\} \tag{38}$$

If we add in the primes determined by the list we get the following sequence of towers and their corresponding numbers.

$$\boldsymbol{F} \boldsymbol{:=}$$

$$\textbf{Expand}\left[\left(1 + \textbf{Total}\left[\left(\boldsymbol{x_1}\textbf{\^{}List@@}\left(\textbf{Expand}\left[\left(1 + \boldsymbol{x_1}\right)\left(1 + \boldsymbol{x_2}\right)\right]\right)\right)\right]\right)\left(1 + \textbf{Total}\left[\left(\boldsymbol{x_2}\textbf{\^{}List@@}\left(1 + \boldsymbol{x_1}\right)\right)\right]\right)\right.$$

$$\left.\left(1 + \boldsymbol{x_3}\right)\left(1 + \boldsymbol{x_4}\right)\right]$$

if we list the resulting towers we get

$$\textbf{List@@}\boldsymbol{F} =$$

$$\{1, x_1, x_1^{x_1}, x_1^{x_2}, x_1^{x_1 x_2}, x_2, x_1 x_2, x_1^{x_1} x_2, x_1^{x_2} x_2, x_1^{x_1 x_2} x_2, x_2^{x_1}, x_1 x_2^{x_1},$$

$$x_1^{x_1} x_2^{x_1}, x_1^{x_2} x_2^{x_1}, x_1^{x_1 x_2} x_2^{x_1}, x_3, x_1 x_3, x_1^{x_1} x_3, x_1^{x_2} x_3, x_1^{x_1 x_2} x_3, x_2 x_3, x_1 x_2 x_3,$$

$$x_1^{x_1} x_2 x_3, x_1^{x_2} x_2 x_3, x_1^{x_1 x_2} x_2 x_3, x_2^{x_1} x_3, x_1 x_2^{x_1} x_3, x_1^{x_1} x_2^{x_1} x_3, x_1^{x_2} x_2^{x_1} x_3,$$

$$x_1^{x_1 x_2} x_2^{x_1} x_3, x_4, x_1 x_4, x_1^{x_1} x_4, x_1^{x_2} x_4, x_1^{x_1 x_2} x_4, x_2 x_4, x_1 x_2 x_4, x_1^{x_1} x_2 x_4,$$

$$x_1^{x_2} x_2 x_4, x_1^{x_1 x_2} x_2 x_4, x_2^{x_1} x_4, x_1 x_2^{x_1} x_4, x_1^{x_1} x_2^{x_1} x_4, x_1^{x_2} x_2^{x_1} x_4, x_1^{x_1 x_2} x_2^{x_1} x_4$$

$$, x_3 x_4, x_1 x_3 x_4, x_1^{x_1} x_3 x_4, x_1^{x_2} x_3 x_4, x_1^{x_1 x_2} x_3 x_4, x_2 x_3 x_4, x_1 x_2 x_3 x_4,$$

$$x_1^{x_1} x_2 x_3 x_4, x_1^{x_2} x_2 x_3 x_4, x_1^{x_1 x_2} x_2 x_3 x_4, x_2^{x_1} x_3 x_4,$$

$$x_1 x_2^{x_1} x_3 x_4, x_1^{x_1} x_2^{x_1} x_3 x_4, x_1^{x_2} x_2^{x_1} x_3 x_4, x_1^{x_1 x_2} x_2^{x_1} x_3 x_4\} \tag{39}$$

The corresponding list of integer is determined by the following Mathematica commands

$$\mathbf{L1} = \mathbf{Sort}\left[\mathbf{List@@}F/.x_1 \to 2/.x_2 \to 3/.x_3 \to 5/.x_4 \to 7\right]$$

which yields the following list of integers.

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 18, 20, 21, 24, 28, 30, 35,$$

$$36, 40, 42, 45, 56, 60, 63, 64, 70, 72, 84, 90, 105, 120, 126, 140, 168, 180,$$

$$192, 210, 252, 280, 315, 320, 360, 420, 448, 504, 576, 630, 840$$

$$, 960, 1260, 1344, 2240, 2520, 2880, 4032, 6720, 20160\} \tag{40}$$

We note that the simple insertion of these towers determines the prime number 11 and 13. The next step is to discuss a slight modification of the straightforward technique discussed above . We see from the algorithm that we are very strongly incentives to reduce the number of terms appearing in the expression for instance it is clear that the number should not be included so we subs-tract it and consider the following expression .

$$A := 1 + \mathbf{Total}\left[(x_1{}^\wedge\mathbf{List@@}\left(\mathbf{Expand}\left[(1 + x_1)(1 + x_2)\right] - x_1 x_2\right))\right]$$

The previous would correspond to what I would refer to as a renormalization step

$$B := 1 + x_2$$

The list of towers and the corresponding list of numbers generated by the reduced product is given by performing by re-normalizing as follows

$$\mathbf{Expand}[AB] - x_1^{x_1} x_2 - x_1^{x_2} x_2$$

$$= 1 + x_1 + x_1^{x_1} + x_1^{x_2} + x_2 + x_1 x_2 \tag{41}$$

$$\mathbf{Sort}\left[\mathbf{List@@}\left(\mathbf{Expand}[AB] - x_1^{x_1} x_2 - x_1^{x_2} x_2\right)/.x_1 \to 2/.x_2 \to 3\right]$$

$$= \{1, 2, 3, 4, 6, 8\} \tag{42}$$

Now if we want to improve the estimate of the sum of the primes in the range and than we must remove from the sum the towers associated with the number in the range of 1 and as follows

$$\mathbf{Expand}\left[(1 + \mathbf{Total}\left[(x_1{}^\wedge\mathbf{List@@}\left(\mathbf{Expand}\left[(1 + x_1)(1 + x_2)\right]\right))\right]) - x_1^{x_1 x_2}\right)(1 + x_2)\right] - x_1^{x_1} x_2 -$$

$$x_1^{x_2} x_2 - 1 - x_1 - x_2$$

$$= x_1^{x_1} + x_1^{x_2} + x_1 x_2 \tag{43}$$

Here is now the estimate if the sum of the primes in the range , is given by the following

$$G := \mathbf{Expand}\left[AB(1 + x_3)(1 + x_4)\right]$$

$$\mathbf{L2} = \mathbf{Sort}\left[\mathbf{List@@}G/.x_1 \to 2/.x_2 \to 3/.x_3 \to 5/.x_4 \to 7\right]$$

$$= \{1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 14, 15, 20, 21, 24, 28, 30, 35, 40,$$

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$$42, 56, 60, 70, 84, 105, 120, 140, 168, 210, 280, 420, 840\} \tag{44}$$

We note that the impact of the reduction is significant we have almost reduced the length we are considering by a factor of 2 as depicted bellow which in of itself is pretty amazing.

$$\textbf{Dimensions[L1]}$$

$$= \{60\} \tag{45}$$

$$\textbf{Dimensions[L2]}$$

$$\{32\} \tag{46}$$

if one just wants to generate the trees without any concern for the the renormalization business here is how one proceeds.

$$\boldsymbol{G_0{:}{=}1}$$

$$\textbf{For}\,[k = 1, k < 3, k{+}{+}, G_0 = \textbf{Expand}\,[G_0\,(1 + x_k)]]$$

$$\boldsymbol{G_1}{:}{=}\textbf{1For}\,[k = 1, k < 3, k{+}{+}, G_1 = \textbf{Expand}\,[G_1\,(1 + \textbf{Total}\,[(x_k{\hat{\;}}\textbf{List@@}\,(G_0))])]]\boldsymbol{G_1}$$

$$= \{1 + x_1 + x_1^{x_1} + x_1^{x_2} + x_1^{x_1 x_2} + x_2 + x_1 x_2 + x_1^{x_1} x_2 + x_1^{x_2} x_2 +$$

$$x_1^{x_1 x_2} x_2 + x_2^{x_1} + x_1 x_2^{x_1} + x_1^{x_1} x_2^{x_1} + x_1^{x_2} x_2^{x_1} + x_1^{x_1 x_2} x_2^{x_1} + x_2^{x_2} + x_1 x_2^{x_2} +$$

$$x_1^{x_1} x_2^{x_2} + x_1^{x_2} x_2^{x_2} + x_1^{x_1 x_2} x_2^{x_2} + x_2^{x_1 x_2} + x_1 x_2^{x_1 x_2} + x_1^{x_1} x_2^{x_1 x_2} + x_1^{x_2} x_2^{x_1 x_2} + x_1^{x_1 x_2} x_2^{x_1 x_2}\}$$

$$\tag{47}$$

which results in the following list of integers.

$$\{1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 27, 36, 54, 64, 72, 108, 192$$

$$, 216, 576, 729, 1458, 1728, 2916, 5832, 46656\} \tag{48}$$

The preceding sequence of numbers allowed us to determine that the primes 5 and 7 are missing from the list.

## 1.4  Recovering the Ordering.

What we would presumably like would be to recover the ordering of the integers by simply manipulating the tower expansion. This would also mean that we would recover the values of the primes. this would not be an easy task but it has the merit of showing how in a sense all the primes are determined by the first two integers. We therefore define

$$g_{s_i}\,(x_{i,t}) = 1 + x_{i,t} + x_{i,t}{}^{x_{i,t}} + \;\cdots\; + \left(x_{i,t}{}^{x_{i,t}{}^{\cdot^{\cdot^{\cdot x_{i,t}}}}}\right\} \text{last term of height } s_i\right) \tag{49}$$

$$G_0\,(\boldsymbol{X}) = \prod_{1 \leq i \leq \dim\{\boldsymbol{X}\}} (1 + g_{s_i}\,(x_{i,1})) \tag{50}$$

We then define the following recurrence

$$G_m\left(\boldsymbol{X}\right) = \prod_{1\leq i\leq \dim\{\boldsymbol{X}\}} \left(1 + \Re\left(x_{i,m},\; G_{m-1}\left(\boldsymbol{X}\right)\right)\right) \tag{51}$$

The basic idea is to perform substitution of the variables in the towers so as to end up with polynomials in the single variable $x$ which can in turn be ordered by recovering the binary decimal expansion corresponding to the numbers. Note that the polynomials obtained when evaluated at $x = 2$ give the value of the integers that the number represents. Let us illustrate the computation for $\boldsymbol{X} = (x_1, x_2)$. Here are the Mathematica Commands illustrating the overall approach

$$\boldsymbol{G_0} = \textbf{Expand}\left[(1 + \boldsymbol{x_{1,1}})(1 + \boldsymbol{x_{2,1}})\right]$$

$$= 1 + x_{1,1} + x_{2,1} + x_{1,1}x_{2,1} \tag{52}$$

$$\textbf{Expand}\left[(1 + \boldsymbol{x_{1,1}})(1 + \boldsymbol{x_{2,1}})\right]$$

$$1 + x_{1,1} + x_{2,1} + x_{1,1}x_{2,1} \tag{53}$$

$$\boldsymbol{G_1} =$$

$$\textbf{Expand}\left[(1 + \textbf{Total}\left[(\boldsymbol{x_{1,2}}\text{\textasciicircum}\textbf{List@@}\left(\textbf{Expand}\left[(1 + \boldsymbol{x_{1,1}})(1 + \boldsymbol{x_{2,1}})\right] - \boldsymbol{x_{1,1}x_{2,1}} - \boldsymbol{x_{2,1}}\right))\right])\right.$$

$$\left.(1 + \boldsymbol{x_{2,1}})\right] - x_{1,2}^{x_{1,1}}x_{2,1}$$

$$= 1 + x_{1,2} + x_{1,2}^{x_{1,1}} + x_{2,1} + x_{1,2}x_{2,1} \tag{54}$$

note that we have performed the renormalization in the expressions above to obtain the following polynomial expressions.

$$\textbf{Expand}\left[\textbf{List@@}\boldsymbol{G_0}/.\boldsymbol{x_{1,1}} \to 2/.\boldsymbol{x_{1,2}} \to \boldsymbol{x}/.\boldsymbol{x_{2,1}} \to 1 + \boldsymbol{x}\right] \tag{55}$$

$$= \left\{1, x, x^2, 1 + x, x + x^2\right\} \tag{56}$$

The expression $G_1$ determines all the primes between $x^2$ and $x^2 + x$ so as to determine the list of consecutive integers from 1 to $x^2 + x$ it suffice to add $x_{3,1}$ to the $G_1$ as follows

$$\boldsymbol{T_1} = \boldsymbol{G_1} + \boldsymbol{x_{3,1}}$$

$$= 1 + x_{1,2} + x_{1,2}^{x_{1,1}} + x_{2,1} + x_{1,2}x_{2,1} + x_{3,1} \tag{57}$$

$$\textbf{Expand}\left[\textbf{List@@}\boldsymbol{T_1}/.\boldsymbol{x_{1,1}} \to 2/.\boldsymbol{x_{1,2}} \to \boldsymbol{x}/.\boldsymbol{x_{2,1}} \to 1 + \boldsymbol{x}/.\boldsymbol{x_{3,1}} \to \boldsymbol{x^2} + 1\right]$$

$$= \left\{1, x, x^2, 1 + x, x + x^2, 1 + x^2\right\} \tag{58}$$

$$\boldsymbol{R_2} = \textbf{Expand}\left[(1 + \textbf{Total}\left[(\boldsymbol{x_{1,2}}\text{\textasciicircum}\textbf{List@@}\left(\textbf{Expand}\left[(1 + \boldsymbol{x_{1,1}})(1 + \boldsymbol{x_{2,1}})\right] - \boldsymbol{x_{1,1}x_{2,1}}\right))\right])\right.$$

$$\left.(1 + \textbf{Total}\left[(\boldsymbol{x_{2,2}}\text{\textasciicircum}\textbf{List@@}\left(\textbf{Expand}\left[(1 + \boldsymbol{x_{1,1}})(1 + \boldsymbol{x_{2,1}})\right] - \boldsymbol{x_{1,1}x_{2,1}} - \boldsymbol{x_{2,1}}\right))\right])(1 + \boldsymbol{x_{3,1}})\right]$$

$$1 + x_{1,2} + x_{1,2}^{x_{1,1}} + x_{1,2}^{x_{2,1}} + x_{2,2} + x_{1,2}x_{2,2} + x_{1,2}^{x_{1,1}}x_{2,2} + x_{1,2}^{x_{2,1}}x_{2,2} + x_{2,2}^{x_{1,1}} +$$

$$x_{1,2}x_{2,2}^{x_{1,1}} + x_{1,2}^{x_{1,1}}x_{2,2}^{x_{1,1}} + x_{1,2}^{x_{2,1}}x_{2,2}^{x_{1,1}} + x_{3,1} + x_{1,2}x_{3,1} + x_{1,2}^{x_{1,1}}x_{3,1} +$$

$$x_{1,2}^{x_{2,1}}x_{3,1} + x_{2,2}x_{3,1} + x_{1,2}x_{2,2}x_{3,1} + x_{1,2}^{x_{1,1}}x_{2,2}x_{3,1} + x_{1,2}^{x_{2,1}}x_{2,2}x_{3,1} +$$

$$x_{2,2}^{x_{1,1}}x_{3,1} + x_{1,2}x_{2,2}^{x_{1,1}}x_{3,1} + x_{1,2}^{x_{1,1}}x_{2,2}^{x_{1,1}}x_{3,1} + x_{1,2}^{x_{2,1}}x_{2,2}^{x_{1,1}}x_{3,1} \tag{59}$$

$$G_2 = R_2 - \left(x_{1,2}^{x_{1,1}} x_{3,1} + x_{1,2}^{x_{2,1}} x_{3,1} + x_{2,2} x_{3,1} + x_{1,2} x_{2,2} x_{3,1} + \right.$$

$$x_{1,2}^{x_{1,1}} x_{2,2} x_{3,1} + x_{1,2}^{x_{2,1}} x_{2,2} x_{3,1} + x_{1,2}^{x_{2,1}} x_{2,2} + x_{1,2} x_{2,2}^{x_{1,1}} x_{3,1} +$$

$$\left. x_{1,2}^{x_{1,1}} x_{2,2}^{x_{1,1}} x_{3,1} + x_{1,2}^{x_{2,1}} x_{2,2}^{x_{1,1}} x_{3,1} + x_{2,2}^{x_{1,1}} x_{3,1} + x_{1,2}^{x_{1,1}} x_{2,2}^{x_{1,1}} + x_{1,2}^{x_{2,1}} x_{2,2}^{x_{1,1}} + x_{1,2} x_{2,2}^{x_{1,1}} \right)$$
$$\tag{60}$$

$$= 1 + x_{1,2} + x_{1,2}^{x_{1,1}} + x_{1,2}^{x_{2,1}} + x_{2,2} + x_{1,2} x_{2,2} + x_{1,2}^{x_{1,1}} x_{2,2} + x_{2,2}^{x_{1,1}} + x_{3,1} + x_{1,2} x_{3,1} \tag{61}$$

$$\text{Expand}\left[\text{List@@}G_2 /.x_{1,2} \to x /.x_{1,1} \to 2\right.$$

$$\left. /.x_{2,1} \to 3 /.x_{2,2} \to (1+x) /.x_{3,1} \to (x^2+1)\right]$$

$$= \left\{1, x, x^2, x^3, 1+x, x+x^2, x^2+x^3, 1+2x+x^2, 1+x^2, x+x^3\right\} \tag{62}$$

$$\text{Sort}\left[\text{Expand}\left[\text{List@@}G_2 /.x_{1,2} \to x /.x_{1,1} \to 2 /.x_{2,1} \to 3\right.\right.$$

$$\left.\left. /.x_{2,2} \to (1+x) /.x_{3,1} \to (x^2+1)\right] /.x \to 2\right]$$

$$= \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12\} \tag{63}$$

$$T_2 = G_2 + x_{4,1} + x_{5,1} =$$

$$1 + x_{1,2} + x_{1,2}^{x_{1,1}} + x_{1,2}^{x_{2,1}} + x_{2,2} + x_{1,2} x_{2,2} + x_{1,2}^{x_{1,1}} x_{2,2} + x_{2,2}^{x_{1,1}} + x_{3,1} + x_{1,2} x_{3,1} + x_{4,1} + x_{5,1}$$
$$\tag{64}$$

In the following list we express the tower expansion for the consecutive integers in the range 0 and $p_5$.

$$\text{Expand}\left[\text{List@@}T_1 /.x_{1,2} \to p_1 /.x_{1,1} \to p_1\right.$$

$$\left. /.x_{2,1} \to p_2 /.x_{2,2} \to p_2 /.x_{3,1} \to p_3 /.x_{4,1} \to p_4 /.x_{5,1} \to p_5\right]$$

$$= \left\{1, p_1, p_1^{p_1}, p_1^{p_2}, p_2, p_1 p_2, p_1^{p_1} p_2, p_2^{p_1}, p_3, p_1 p_3, p_4, p_5\right\} \tag{65}$$

In the following list we express the tower expansion for the consecutive integers in the range 0 and $p_5$ in the special polynomial form.

$$\text{Expand}\left[\text{List@@}T_1 /.x_{1,2} \to x /.x_{1,1} \to 2 /.x_{2,1} \to 3 /.x_{2,2} \to (1+x)\right.$$

$$\left. /.x_{3,1} \to (x^2+1) /.x_{4,1} \to x^2+x+1 /.x_{5,1} \to x^3+x^2+1\right]$$

$$= \left\{1, x, x^2, x^3, 1+x, x+x^2, x^2+x^3, 1+2x+x^2, 1+x^2, x+x^3, 1+x+x^2, 1+x^2+x^3\right\}$$
$$\tag{66}$$

One continues the process illustrated above to obtain The polynomial representation of the primes in the sought after range. Furthermore the above expansion suggest a alternative way of representing numbers in the sense that these are reducible polynomials in the field $[\{+, \times\}; \{0, 1\}]$ and the factorization can be recovered in polynomial time. The problems of course is that addition completely messes things up. So in order to recover correct polynomial representation of a given integers after performing one or several addition consists in generating towers in the neighborhood of the sought after polynomials and check that they both evaluate to the same number. Another important fact following from the discussion above is the fact that the arithmetic is being performed in a manner which mimics operation on sets one can think of these operations as operations on forests with rooted trees with colored vertices where each one one of the primes represent a color.

## 1.5 Getting hold of the rationals.

We recall that

$$g(x) = 1 + x + x^x + x^{x^x} + x^{x^{x^x}} + x^{x^{x^{x^x}}} + \cdots \tag{67}$$

Let

$$\boldsymbol{P} \equiv (p_1, \cdots, p_k, \cdots) \tag{68}$$

denote the vectors of the primes.

$$\lim_{n \to \infty} \left\{ H_n(\boldsymbol{P}) = \prod_{1 \leq k \leq |\boldsymbol{P}|} \left( \mathfrak{R}\left(p_k^{-1}, G_n(\boldsymbol{P})\right) + 1 + \mathfrak{R}\left(p_k, G_n(\boldsymbol{P})\right) \right) \right\} \tag{69}$$

The terms in the sums are in bijective correspondence with the element on $\mathbb{Q}$the code for producing rational is given bellow. In what follows we illustrate how to generate towers associated with rational numbers.

$$g_t := 1 + x_t$$

$$\text{For}\,[i = 1, i < 1, i{+}{+}, g_t = (1 + \text{Total}\,[(x_t{}^\wedge \text{List@@}\,(g_t))])]$$

$$G_0 := 1$$

$$\text{For}\,[k = 1, k < 3, k{+}{+}, G_0 = \text{Expand}\,[G_0\,(g_t/.t \to k)]]$$

$$G_1 := 1$$

$$\text{For}\,[k = 1, k < 3, k{+}{+}, G_1 = \text{Expand}\,[G_1\,(1 + \text{Total}\,[(x_k{}^\wedge \text{List@@}\,(G_0))])]]$$

$$H_1 := 1$$

$$\text{For}\,[k = 1, k < 3, k{+}{+},$$

$$H_1 = \text{Expand}\,\left[H_1\,\left(\text{Total}\,\left[\left(x_k{}^{-1}\right){}^\wedge \text{List@@}\,(G_0)\right] + 1 + \text{Total}\,[x_k{}^\wedge \text{List@@}\,(G_0)]\right)\right]]$$

$$H_1$$

$$= 1 + \left(\frac{1}{x_1}\right)^{x_1} + \left(\frac{1}{x_1}\right)^{x_2} + \left(\frac{1}{x_1}\right)^{x_1 x_2} + \frac{1}{x_1} + x_1 + x_1^{x_1} + x_1^{x_2} + x_1^{x_1 x_2} +$$

$$\left(\frac{1}{x_2}\right)^{x_1} + \left(\frac{1}{x_1}\right)^{x_1}\left(\frac{1}{x_2}\right)^{x_1} + \left(\frac{1}{x_1}\right)^{x_2}\left(\frac{1}{x_2}\right)^{x_1} + \left(\frac{1}{x_1}\right)^{x_1 x_2}\left(\frac{1}{x_2}\right)^{x_1} + \frac{\left(\frac{1}{x_2}\right)^{x_1}}{x_1} +$$

$$x_1\left(\frac{1}{x_2}\right)^{x_1} + x_1^{x_1}\left(\frac{1}{x_2}\right)^{x_1} + x_1^{x_2}\left(\frac{1}{x_2}\right)^{x_1} + x_1^{x_1 x_2}\left(\frac{1}{x_2}\right)^{x_1} + \left(\frac{1}{x_2}\right)^{x_2} + \left(\frac{1}{x_1}\right)^{x_1}\left(\frac{1}{x_2}\right)^{x_2} +$$

$$\left(\frac{1}{x_1}\right)^{x_2}\left(\frac{1}{x_2}\right)^{x_2} + \left(\frac{1}{x_1}\right)^{x_1 x_2}\left(\frac{1}{x_2}\right)^{x_2} + \frac{\left(\frac{1}{x_2}\right)^{x_2}}{x_1} + x_1\left(\frac{1}{x_2}\right)^{x_2} + x_1^{x_1}\left(\frac{1}{x_2}\right)^{x_2} + x_1^{x_2}\left(\frac{1}{x_2}\right)^{x_2} +$$

$$x_1^{x_1 x_2}\left(\frac{1}{x_2}\right)^{x_2}+\left(\frac{1}{x_2}\right)^{x_1 x_2}+\left(\frac{1}{x_1}\right)^{x_1}\left(\frac{1}{x_2}\right)^{x_1 x_2}+\left(\frac{1}{x_1}\right)^{x_2}\left(\frac{1}{x_2}\right)^{x_1 x_2}+\left(\frac{1}{x_1}\right)^{x_1 x_2}\left(\frac{1}{x_2}\right)^{x_1 x_2}+$$

$$\frac{\left(\frac{1}{x_2}\right)^{x_1 x_2}}{x_1}+x_1\left(\frac{1}{x_2}\right)^{x_1 x_2}+x_1^{x_1}\left(\frac{1}{x_2}\right)^{x_1 x_2}+x_1^{x_2}\left(\frac{1}{x_2}\right)^{x_1 x_2}+x_1^{x_1 x_2}\left(\frac{1}{x_2}\right)^{x_1 x_2}+\frac{1}{x_2}+\frac{\left(\frac{1}{x_1}\right)^{x_1}}{x_2}+$$

$$\frac{\left(\frac{1}{x_1}\right)^{x_2}}{x_2}+\frac{\left(\frac{1}{x_1}\right)^{x_1 x_2}}{x_2}+\frac{1}{x_1 x_2}+\frac{x_1}{x_2}+\frac{x_1^{x_1}}{x_2}+\frac{x_1^{x_2}}{x_2}+\frac{x_1^{x_1 x_2}}{x_2}+x_2+\left(\frac{1}{x_1}\right)^{x_1}x_2+\left(\frac{1}{x_1}\right)^{x_2}x_2+$$

$$\left(\frac{1}{x_1}\right)^{x_1 x_2}x_2+\frac{x_2}{x_1}+x_1 x_2+x_1^{x_1}x_2+x_1^{x_2}x_2+x_1^{x_1 x_2}x_2+x_2^{x_1}+\left(\frac{1}{x_1}\right)^{x_1}x_2^{x_1}+\left(\frac{1}{x_1}\right)^{x_2}x_2^{x_1}+$$

$$\left(\frac{1}{x_1}\right)^{x_1 x_2}x_2^{x_1}+\frac{x_2^{x_1}}{x_1}+x_1 x_2^{x_1}+x_1^{x_1}x_2^{x_1}+x_1^{x_2}x_2^{x_1}+x_1^{x_1 x_2}x_2^{x_1}+x_2^{x_2}+\left(\frac{1}{x_1}\right)^{x_1}x_2^{x_2}+\left(\frac{1}{x_1}\right)^{x_2}x_2^{x_2}+$$

$$\left(\frac{1}{x_1}\right)^{x_1 x_2}x_2^{x_2}+\frac{x_2^{x_2}}{x_1}+x_1 x_2^{x_2}+x_1^{x_1}x_2^{x_2}+x_1^{x_2}x_2^{x_2}+x_1^{x_1 x_2}x_2^{x_2}+x_2^{x_1 x_2}+\left(\frac{1}{x_1}\right)^{x_1}x_2^{x_1 x_2}+$$

$$\left(\frac{1}{x_1}\right)^{x_2}x_2^{x_1 x_2}+\left(\frac{1}{x_1}\right)^{x_1 x_2}x_2^{x_1 x_2}+\frac{x_2^{x_1 x_2}}{x_1}+x_1 x_2^{x_1 x_2}+x_1^{x_1}x_2^{x_1 x_2}+x_1^{x_2}x_2^{x_1 x_2}+x_1^{x_1 x_2}x_2^{x_1 x_2}$$

$$\tag{70}$$

It should be noted that in the resulting expression the towers associated with rational appear only once in the sequence.

## 1.6 Addition and Subtraction Algorithms.

We assume that numbers are given in their prime tower representation and consider the recursive algorithm expressed by

$$\begin{cases} n+m=\left(\sqrt{n}+\sqrt{m}\right)^2-2\sqrt{n}\cdot\sqrt{m} \\ n-m=\left(\sqrt{n}-\sqrt{m}\right)\cdot\left(\sqrt{n}+\sqrt{m}\right) \end{cases} \tag{71}$$

An alternative option for adding towers include sieving and checking or solving the optimization proble induced by the identity

$$d\left(T_{\boldsymbol{X}}\left(m\right),T_{\boldsymbol{X}}\left(p\right)\right)=d\left(0,T_{\boldsymbol{X}}\left(n\right)\right)\Leftrightarrow T_{\boldsymbol{X}}\left(m\right)+T_{\boldsymbol{X}}\left(n\right)=T_{\boldsymbol{X}}\left(p\right) \tag{72}$$

assuming that we have recovered the ordering for sufficiently many towers we may check for candidate towers $T_{\boldsymbol{X}}\left(p\right)$ for which the following conditions are fulfilled

$$\begin{array}{c} T_{\boldsymbol{X}}\left(p\right)\geq T_{\boldsymbol{X}}\left(n\right) \\ and \\ T_{\boldsymbol{X}}\left(p\right)\geq T_{\boldsymbol{X}}\left(m\right) \end{array} \tag{73}$$

$$\left(T_{\boldsymbol{X}}\left(m\right)+T_{\boldsymbol{X}}\left(n\right)=T_{\boldsymbol{X}}\left(p\right)\right)\Leftrightarrow\begin{cases} d\left(T_{\boldsymbol{X}}\left(m\right),T_{\boldsymbol{X}}\left(p\right)\right)\geq d\left(0,T_{\boldsymbol{X}}\left(n\right)\right) \\ d\left(T_{\boldsymbol{X}}\left(m\right),T_{\boldsymbol{X}}\left(p\right)\right)\leq d\left(0,T_{\boldsymbol{X}}\left(n\right)\right) \end{cases} \tag{74}$$

## 1.7 Functional relation of tower progression

Tower progression are defined by the following expression

$$S_n\left(x\right) = 1 + x + x^x + x^{\left(x^x\right)} + \cdots + \left( \left. x^{\left(x^{\left(\cdot^{\cdot^{\left(x^x\right)}}\right)}\right)} \right\} height = n \right) \qquad (75)$$

from which it follows that

$$\begin{cases} S_{n+1}\left(x\right) & = & 1 + \mathfrak{R}\left(x,\, S_n\left(x\right)\right) \\ \\ S_{n+1}\left(x\right) & = & S_n\left(x\right) + \left( \left. x^{\left(x^{\left(\cdot^{\cdot^{\left(x^x\right)}}\right)}\right)} \right\} height = n \right) \end{cases} \qquad (76)$$

So that tower progression is determined by the following equation

$$1 + \mathfrak{R}\left(x,\, S_n\left(x\right)\right) - S_n\left(x\right) - \left( \left. x^{\left(x^{\left(\cdot^{\cdot^{\left(x^x\right)}}\right)}\right)} \right\} height = n \right) = 0 \qquad (77)$$

# 2 Conclusion

In part I we have proposed an inherently combinatorial approach to investigating properties of numbers.

# References

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